# $\mathrm{Hilb}^n(\mathbb{C}^2)$ and link homology

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \left\{ \text{ideals } I \subset \mathbb{C}[x,y] \;\middle|\; \dim \frac{\mathbb{C}[x,y]}{I} = n \right\}$$

<u>Smooth</u>, $\dim = 2n$, symplectic

$$\begin{array}{ccc} \mathrm{Hilb}^n(\mathbb{C}^2) & \xrightarrow{\pi} & S^n\mathbb{C}^2 \\ I & \longmapsto & \mathrm{supp}\left(\frac{\mathbb{C}[x,y]}{I}\right) \end{array}$$

symplectic resolution! (see Joel's lectures)

$$S^n\mathbb{C}^2 = \mathrm{Spec}\, \mathbb{C}[x_1 \ldots x_n, y_1 \ldots y_n]^{S_n}$$ affine Poisson.

Two $\mathbb{C}^*$ actions on $\mathbb{C}^2$ lift to $\mathbb{C}^*$ actions on $\mathrm{Hilb}^n(\mathbb{C}^2)$:

- Hamiltonian torus $(x,y) \to (qx, q^{-1}y)$ $T$
- Conical torus $(x,y) \longrightarrow (tx, ty)$
- Lagrangian (attracting) subvariety:
  $\lim_{q \to 0} (qx, q^{-1}y)$ exists if $y = 0$

$$\Rightarrow L = \mathrm{Hilb}^n(\mathbb{C}^2)^+ = \mathrm{Hilb}^n(\mathbb{C}^2, \mathbb{C})$$

notations from Joel's lecture

$\mathrm{supp}\,(\mathbb{C}[x,y]) \subset \{y=0\}$

$\mathrm{supp}\left(\frac{\mathbb{C}[x,y]}{I}\right) \subset \{y=0\}$

- $T$-fixed points = monomial ideals

$\lambda =$ partition of $n$


$I_\lambda = (x^6, x^5y, x^2y^2, xy^3, y^4)$

$I_\lambda$ = ideal generated by monomials outside $\lambda$

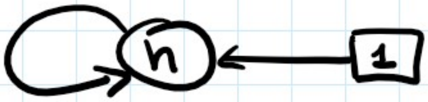
- Tautological bundle $\mathcal{T}$, rank $= n$

fiber over $I = \frac{\mathbb{C}[x,y]}{I}$

$\det \mathcal{T} = \Lambda^n \mathcal{T} = \mathcal{O}(1)$ line bundle

- $\mathrm{Hilb}^n(\mathbb{C}^2)$ = <u>both</u> the Higgs and Coulomb branch

for $(G, N) = (GL_n, \mathfrak{gl}_n \oplus \mathbb{C}^n)$




(affine $A_1$)

(Braverman, Finkelberg, Nakajima)

<u>Conj</u> (G., Negut, Rasmussen) = <u>Thm</u> (Oblomkov, Rozansky)

$\beta$ = braid on $n$ strands $\longrightarrow \mathcal{F}_\beta$ $\mathbb{C}^* \times \mathbb{C}^*$ equivariant sheaf on $\mathrm{Hilb}^n(\mathbb{C}^2, \mathbb{C})$

(a) $\quad HHH(\beta) = H^\bullet(\mathrm{Hilb}^n(\mathbb{C}^2, \mathbb{C}), \mathcal{F}_\beta \otimes \Lambda^\bullet \mathcal{T}^\vee)$

(a) $\mathrm{HHH}(\beta) = H^{\bullet}_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{Hilb}^n(\mathbb{C}^2,\mathbb{C}), \mathcal{F}_\beta \otimes \Lambda^{\bullet} \mathcal{T}^\vee)$

q, t - degrees        a-degree

(b) Action of $x_i$ on $\mathrm{HHH}(\beta) \longleftrightarrow$ support of $\mathcal{F}_\beta$

Ex $\beta$ = knot $\Rightarrow$ all $x_i$ act the same way on $\mathrm{HHH}(\beta)$

$\Rightarrow \mathcal{F}_\beta$ is supported on $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$

all points are the same = 0
up to shift.

Ex $\beta = T(n, kn+1) \Rightarrow \mathcal{F}_\beta = \mathcal{O}(k)$ on $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$

(c) More generally, $\beta \longrightarrow \beta \cdot T(n,n)$   full twist

$\mathcal{F}_\beta \longrightarrow \mathcal{F}_\beta \otimes \mathcal{O}(1)$

Ex $\beta = T(2,3) \Rightarrow \mathcal{F}_\beta = \mathcal{O}(1)$ on $\mathrm{Hilb}^2(\mathbb{C}^2, 0) = \mathbb{P}^1$

$H^0_{\mathbb{C}^*\times\mathbb{C}^*}(\mathbb{P}^1, \mathcal{O}(1)) = q + t$

$\beta = T(3,4) \Rightarrow \mathcal{F}_\beta = \mathcal{O}(1)$ on $\mathrm{Hilb}^3(\mathbb{C}^2, 0) =$ cone over twisted cubic

Exercise $\dim H^0(\mathrm{Hilb}^3(\mathbb{C}^2,0), \mathcal{O}(1)) = 5$

same as $\mathrm{HHH}(T(3,4))$!

Ex $\beta = T(m,n)$

$$\mathrm{FHilb}^n(\mathbb{C}^2,0) \xrightarrow{p} \mathrm{Hilb}^n(\mathbb{C}^2,0)$$

$$FHilb^n(\mathbb{C}^2,0) \xrightarrow{p} Hilb^n(\mathbb{C}^2,0)$$

$$\overset{\shortparallel}{\{\mathbb{C}[x,y] \supset I_1 \supset \dots \supset I_n\}}$$

nested Hilbert scheme

$\mathcal{L}_k = I_{k-1}/I_k$ line bundle

$$\mathcal{F}_{T(m,n)} = p_*(\mathcal{L}_1^{a_1} \cdots \mathcal{L}_n^{a_n}) \quad \text{(G.–Negut)}$$

where $a_i = \lfloor \frac{im}{n} \rfloor - \lfloor \frac{(i-1)m}{n} \rfloor$

Note: Need to use dg structure on FHilb to define $p_*$.

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What about identity braid?

$$\begin{array}{ccc} X_n & \longrightarrow & (\mathbb{C}^2)^n \\ q\downarrow & & \downarrow \\ Hilb^n(\mathbb{C}^2) & \longrightarrow & S^n\mathbb{C}^2 \end{array}$$

Thm (Haiman) (a) $X_n$ = blow up of $(\mathbb{C}^2)^n$ along the union of diagonals

(b) $X_n = \mathrm{Proj} \bigoplus_{k=0}^{\infty} J^k$ where

$J = \bigcap_{i \neq j} (x_i - x_j, y_i - y_j)$ = ideal of the union of diagonals.

(c) $p_* \mathcal{O}_{X_n} = \mathcal{P}$ Procesi bundle

$\mathcal{P}$ = vector bundle on $Hilb^n(\mathbb{C}^2)$

$P$ = vector bundle on $\mathrm{Hilb}^n(\mathbb{C}^2)$

(d)
$$\begin{tikzcd}
\underset{\{I_1 \supset I_2 \supset \dots \supset I_n\}}{F\mathrm{Hilb}^n(\mathbb{C}^2)} \arrow[r] \arrow[rrr, bend left, "\{\mathrm{supp}(I_{k-1}/I_k)\}"] \arrow[dr, bend right, "p", "I_n"'] & X_n \arrow[rr] \arrow[d, "q"] & & (\mathbb{C}^2)^n \arrow[d] \\
& \mathrm{Hilb}^n(\mathbb{C}^2) \arrow[rr] & & S^n\mathbb{C}^2
\end{tikzcd}$$

$$p_* \mathcal{O}_{F\mathrm{Hilb}^n(\mathbb{C}^2)} = q_* \mathcal{O}_{X_n} = P$$

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$$\beta = \mathbb{1}\ |||\ldots| \rightsquigarrow \mathcal{F}_\beta = P\big|_{\mathrm{Hilb}^n(\mathbb{C}^2,\mathbb{C})}$$

$$\beta = T(n, kn) \rightsquigarrow \mathcal{F}_\beta = P \otimes \mathcal{O}(k)\big|_{\mathrm{Hilb}^n(\mathbb{C}^2,\mathbb{C})}$$

**Note:** $H^0(\mathrm{Hilb}^n(\mathbb{C}^2,\mathbb{C}), P \otimes \mathcal{O}(k))$

$$= H^0(X_n(\mathbb{C}^2,\mathbb{C}), \mathcal{O}(k)) =$$

$$= J^k / (y) J^k$$

since $H^0(X_n, \mathcal{O}(k)) = J^k$ and $J^k$ is free over $\mathbb{C}[y_1 \dots y_n]$

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Approaches to proof / construction of $\mathcal{F}_\beta$

① (Oblomkov–Rozansky) [illegible]

① (Oblomkov–Rozansky) Define a link invariant using MF on $\approx$ FHilb$^n$, then prove it is isomorphic to HHH.

② (G.–Hogancamp–Wedrich) Use derived categorical traces to study link homology in the solid torus.

③ (G.–Negut–Rasmussen)

Consider the graded algebra

$$A = \bigoplus_{k \geq 0} \mathrm{Hom}(R, FT^k)$$ where $FT$ = full twist braid $= T(n,n)$

Multiplication: $\qquad FT^k = T(n, kn)$

$$\mathrm{Hom}(R, FT^k) \otimes \mathrm{Hom}(R, FT^\ell) \to \mathrm{Hom}(R, FT^{k+\ell})$$

<u>Recall</u> (Lecture 3)

$$\mathrm{Hom}(R, FT^k) = HHH^0(FT^k)$$

$$= J^k / (y) J^k$$

where $J = \bigcap_{i \neq j} (x_i - x_j, y_i - y_j)$ as above!

Given a braid $\beta \rightsquigarrow T_\beta$ complex of bimodules

Given a braid $\beta \rightsquigarrow T_\beta$ complex of $A$ bimodules

$\rightsquigarrow \widehat{\bigoplus_{k=0}} \mathrm{Hom}(R, T_\beta \cdot FT^k)$ = graded module over $A$ =

coherent sheaf on $\boxed{\mathrm{Proj}\, A = X_n(\mathbb{C}^2, \mathbb{C})}$

Expect this to lift to a (dg) functor ...

$K^b(SBim_n) \longrightarrow D^b(\mathrm{Hilb}^n(\mathbb{C}^2, \mathbb{C}))$

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# THANK YOU!

Want to learn more about link homology?

AIM research community

https://aimath.org/programs/researchcommunities/linkhom/